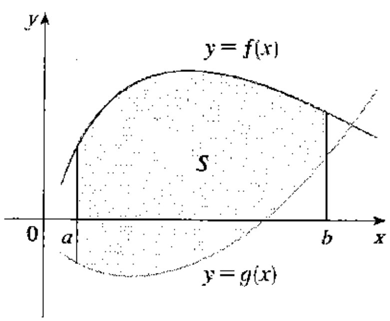
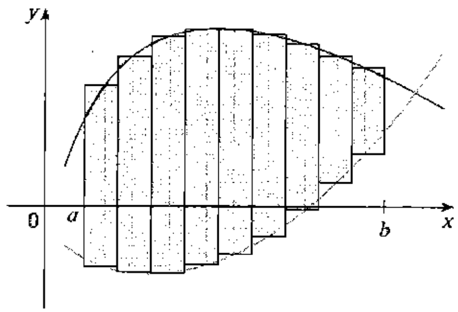
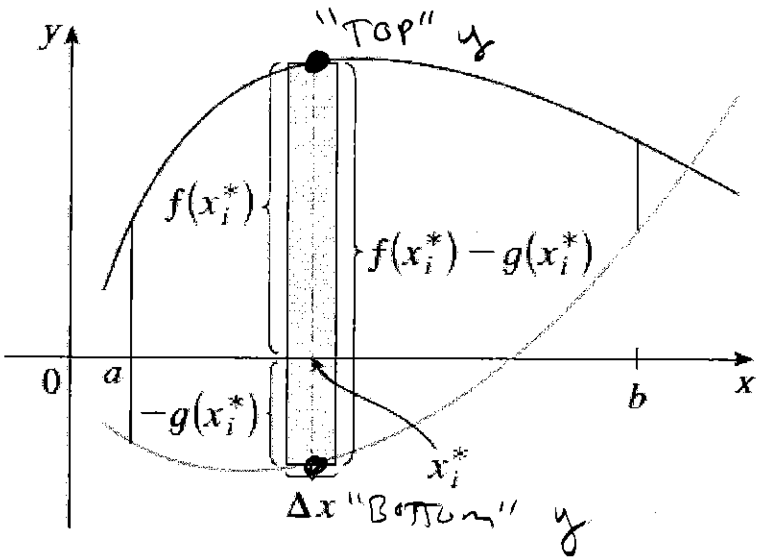
Ch 6: Basic Integral Applications

6.1 Areas Between Curves

Using dx:







(a) Typical rectangle

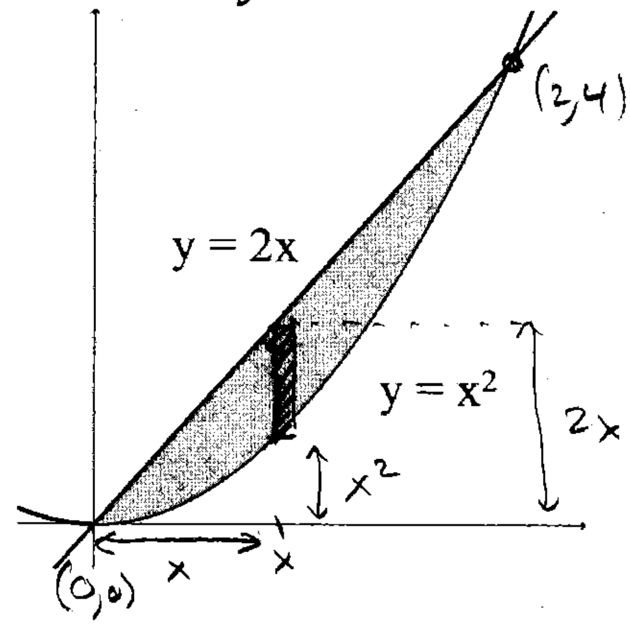
$$\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} (f(x_i) - g(x_i))\Delta x$$

Example: Find the area bounded between $y = 2x$ and $y = x^2$.

$2x = x^2$

$\Rightarrow 0 = x^2 - 2x$

$0 = x(x-2)$



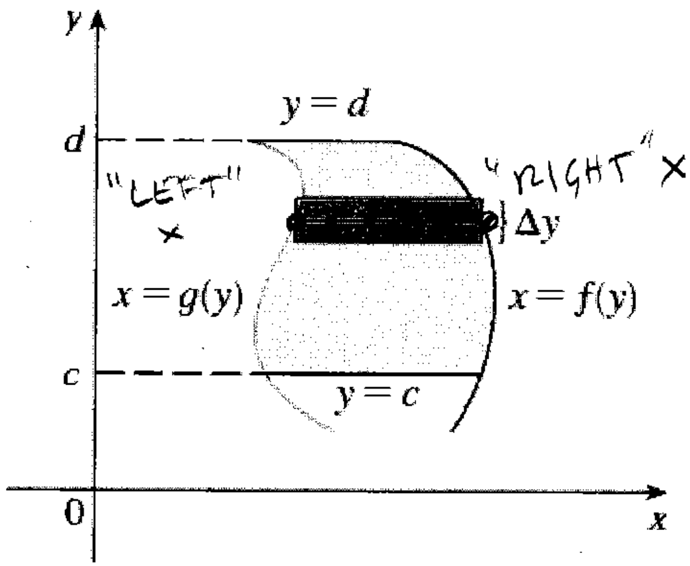
$$\int_0^2 2x - x^2 \, dx$$

$$x^2 - \frac{1}{3}x^3 \Big|_0^2 = \left((2)^2 - \frac{1}{3}(2)^3\right) - \left(0^2 - \frac{1}{3}(0)^3\right)$$

$$= 4 - 8/3$$

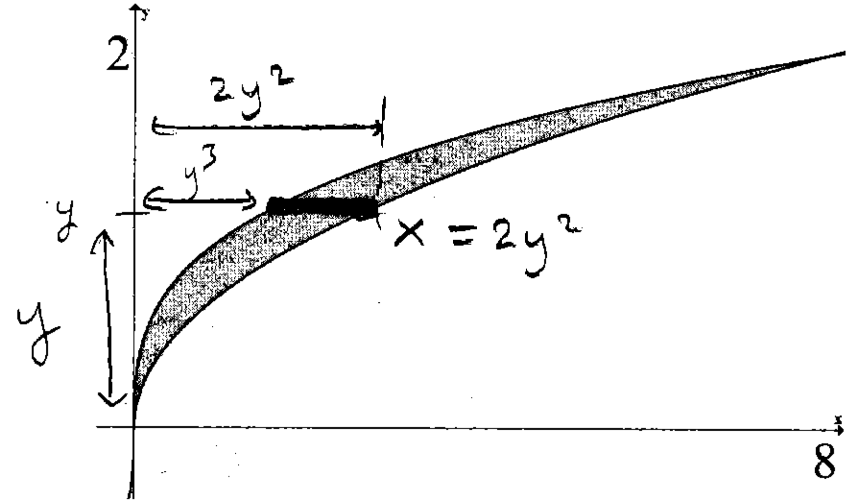
$$= \boxed{4/3}$$

Using dy:



$$\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} (f(y_i) - g(y_i))\Delta y$$

Example: Set up an integral for the area bounded between $x = 2y^2$ and $x = y^3$ (shown below) using dy.



$$\int_0^2 2y^2 - y^3 \, dy$$

$$= \frac{2}{3}y^3 - \frac{1}{4}y^4 \Big|_0^2$$

$$= \left(\frac{2}{3}(2)^3 - \frac{1}{4}(2)^4\right) - 0$$

$$= \frac{16}{3} - 4 = \boxed{4/3}$$

Summary: The area between curves

1. Draw picture finding all intersections.
2. Choose dx or dy. Get ***everything*** in terms of the variable you choose.
3. Draw a typical approx. rectangle.
4. Set up as follows:

$$\text{Area} = \int_a^b (\text{TOP} - \text{BOTTOM})\, dx$$

$$\text{Area} = \int_c^d (\text{RIGHT} - \text{LEFT})\, dy$$

HERE IS WHAT IT WOULD LOOK LIKE

$$\underbrace{\int_0^1 \sqrt{x} - -\sqrt{x}\, dx + \int_1^4 \sqrt{x} - (x-2)\, dx}$$

ALSO CORRECT

Example: Set up an integral (or integrals) that give the area of the region bounded by x = y^2 and y = x – 2.

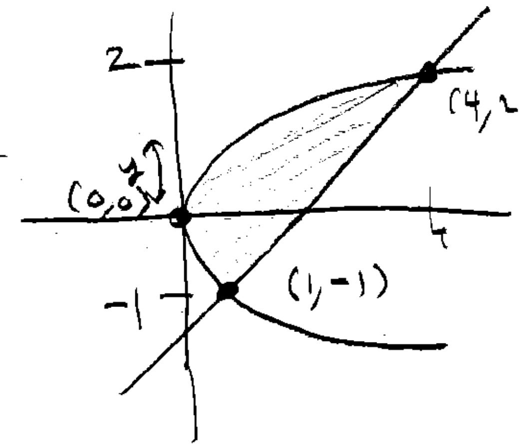
$x = y^2 \Leftrightarrow \begin{cases} y = \sqrt{x} \\ y = -\sqrt{x} \end{cases}$

$x = y + 2 \Leftrightarrow y = x - 2$

$y^2 = y + 2$

$y^2 - y - 2 = 0$

$(y-2)(y+1) = 0$



DON'T USE X! $\left\{ \begin{array}{l} \text{BOTTOM CHANGES} \\ \text{AT } x = 1 \end{array} \right.$

RIGHT − LEFT

$$\int_{-1}^{2} y + 2 - y^2\, dy$$

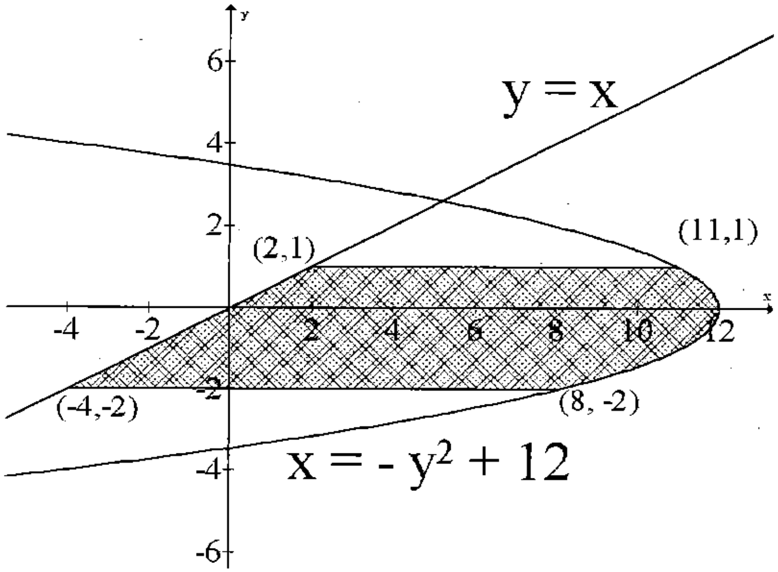
$$\frac{1}{2}y^2 + 2y - \frac{1}{3}y^3 \Big|_{-1}^{2}$$

$$\left(\frac{1}{2}(2)^2 + 2(2) - \frac{1}{3}(2)^3\right) - \left(\frac{1}{2}(-1)^2 + 2(-1) - \frac{1}{3}(-1)^3\right)$$

$$\left(2 + 4 - 8/3\right) - \left(\frac{1}{2} - 2 + 1/3\right)$$

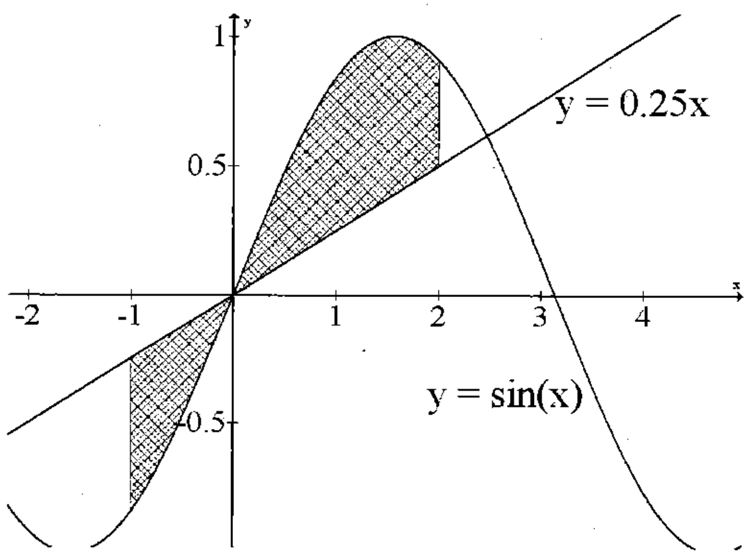
$$8 - 9/3 - 1/2 = 5 - 1/2 = \boxed{9/2}$$

Set up an integral for the total positive area of the following regions:



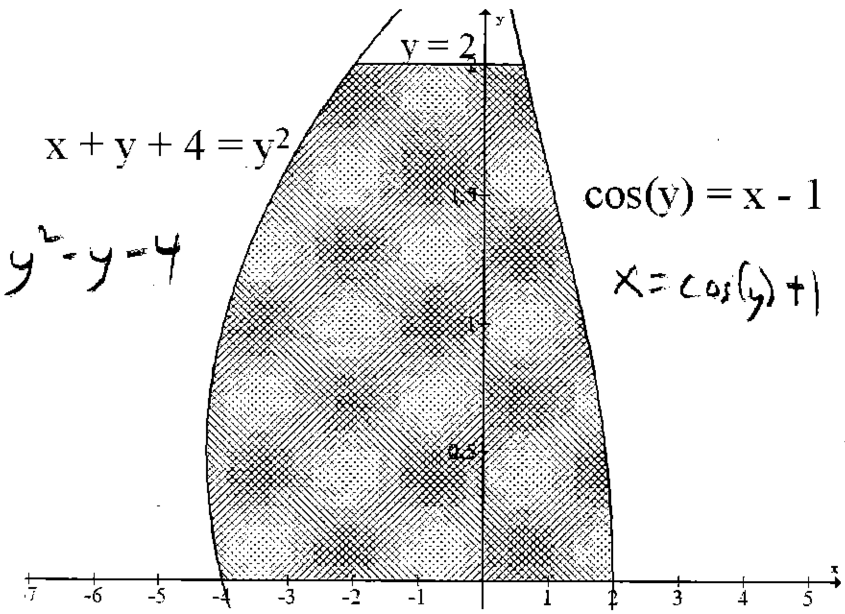
USE dy!!!

$$\int_{-2}^{2} (-y^2+12) - y \, dy$$



USE dx!!!

$$\int_{-1}^{0} \frac{1}{4}x - \sin(x) \, dx + \int_{0}^{2} \sin(x) - \frac{1}{4}x \, dx$$

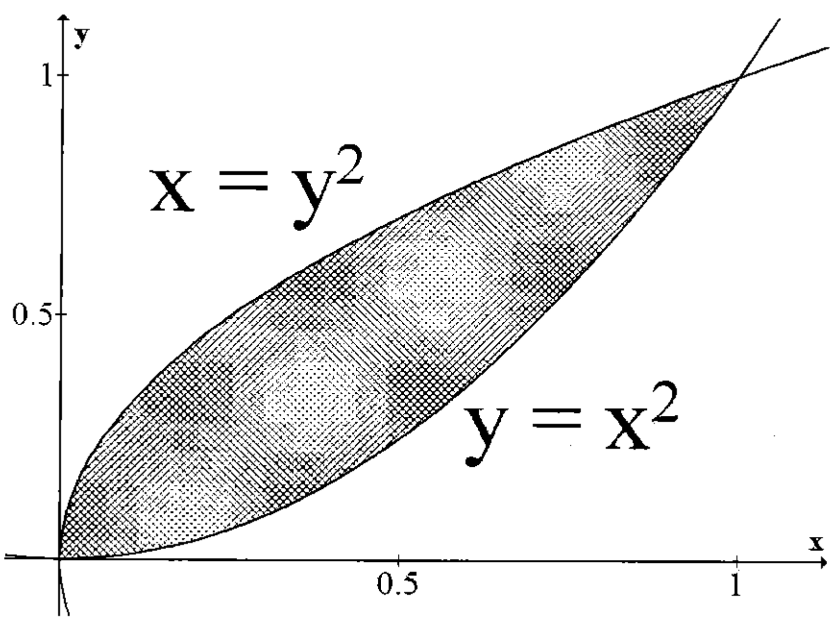


$x = y^2 - y - 4$

$x = \cos(y) + 1$

USE dy!!!

$$\boxed{\int_0^2 (\cos(y)+1) - (y^2-y-4)\,dy}$$

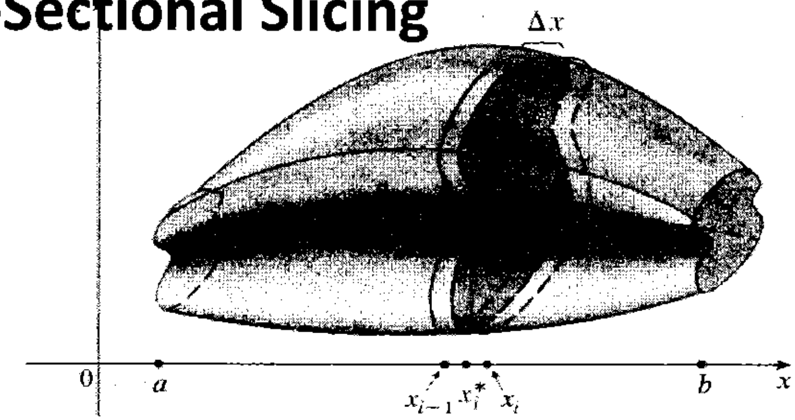


BOTH WORK WELL

dx: $\boxed{\int_0^1 \sqrt{x} - x^2\,dx}$

dy: $\boxed{\int_0^1 \sqrt{y} - y^2\,dy}$

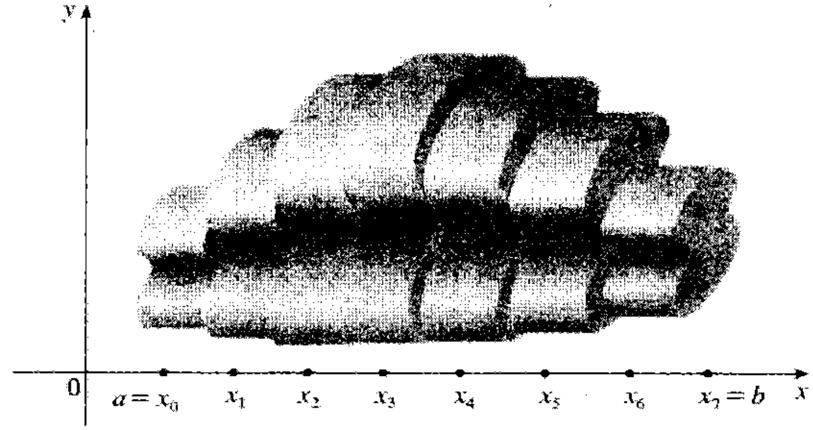
5.2 Finding Volumes Using Cross-Sectional Slicing



If we can find the general formula, $A(x_i)$, for the area of a cross-sectional slice, then we can approximate volume by:

$$\text{Volume of one slice } \approx \mathrm{A}(\mathrm{x}_i)\,\Delta\mathrm{x}$$

$$\text{Total Volume } \approx \sum_{i=1}^{n} \mathrm{A}(\mathrm{x}_i)\,\Delta\mathrm{x}$$



This approximation gets better and better with more subdivisions, so

$$\text{Exact Volume} = \lim_{n\to\infty} \sum_{i=1}^{n} \mathrm{A}(\mathrm{x}_i)\,\Delta\mathrm{x}$$

We conclude

$$\text{Volume } = \int_a^b A(x)dx =$$

$$\int_a^b \text{"Cross-sectional area formula"}\,dx$$

Volume using cross-sectional slicing

1. Draw region. Cut **perpendicular** to rotation axis. Label *x* if that cut crosses the *x*-axis (and *y* if *y*-axis). Label **everything** in terms this variable.

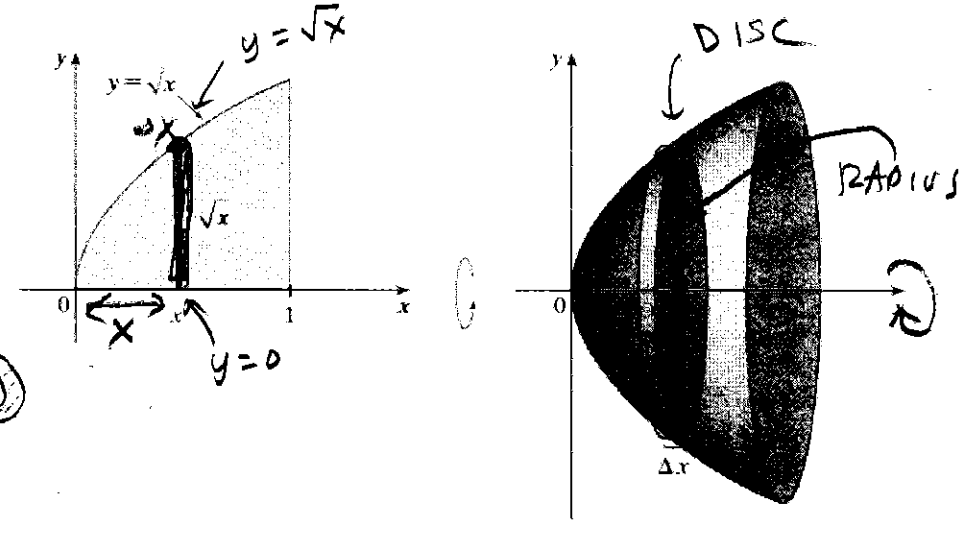
2. Formula for cross-sectional area?

disc:	Area = $\pi(\text{radius})^2$
washer:	Area = $\pi(\text{outer})^2 - \pi(\text{inner})^2$
square:	Area = (Height)(Length)
triangle:	Area = ½ (Height)(Length)

SAME

3. Integrate the area formula.

Example: Consider the region, *R*, bounded by $y = \sqrt{x}$, y = 0, and x = 1. Find the volume of the solid obtained by rotating R about the **x-axis**.



(a) (b)

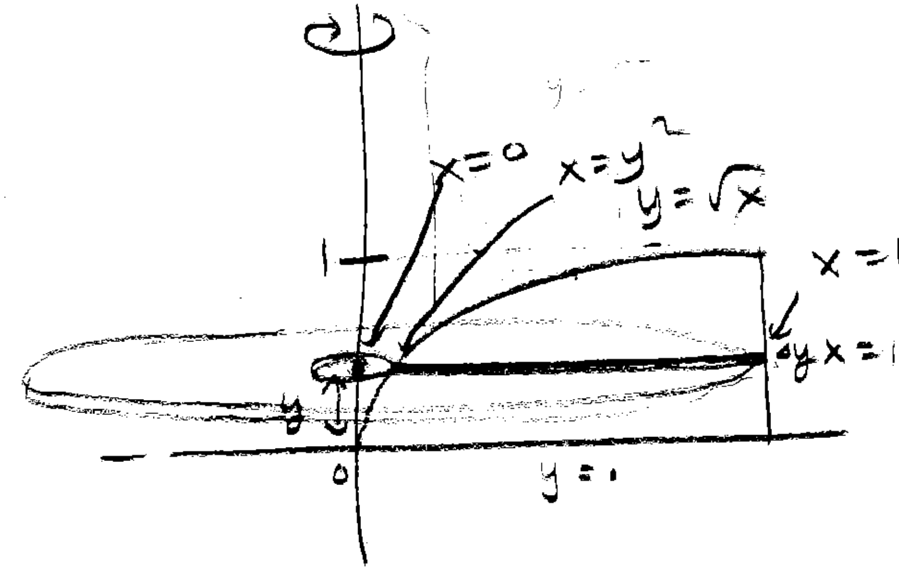
$$\int_0^1 \pi (\text{RADIUS})^2 \, dx$$

$$= \int_0^1 \pi (\sqrt{x})^2 \, dx$$

$$= \pi \int_0^1 x \, dx = \pi \, \frac{1}{2} x^2 \Big|_0^1$$

$$= \boxed{\pi/2}$$

Example: Consider the region, R, bounded by $y = \sqrt{x}$, y = 0, and x = 1. Find the volume of the solid obtained by rotating R about the **y-axis**.

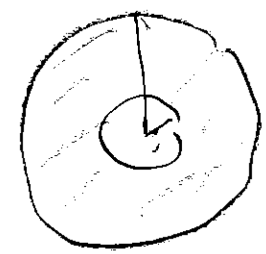


$$\int_0^1 \pi(\ 1\ )^2 - \pi\,(y^2)^2\,dy$$

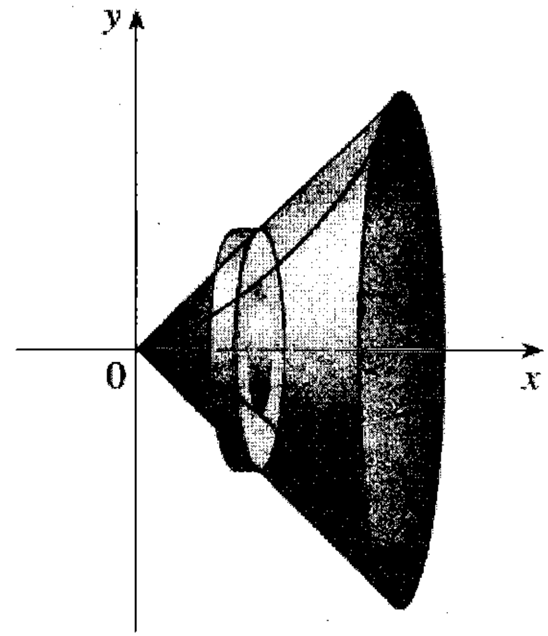
$$= \pi \int_0^1 1 - y^4\,dy$$

$$= \pi\left(y - \frac{1}{5}y^5\Big|_0^1\right)$$

$$= \pi\left(1 - \tfrac{1}{5}\right) = \boxed{4\pi/5}$$



Example: Consider the region, R, bounded by $y = x$ and $y = x^4$. Find the volume of the solid obtained by rotating R about the **x-axis**.

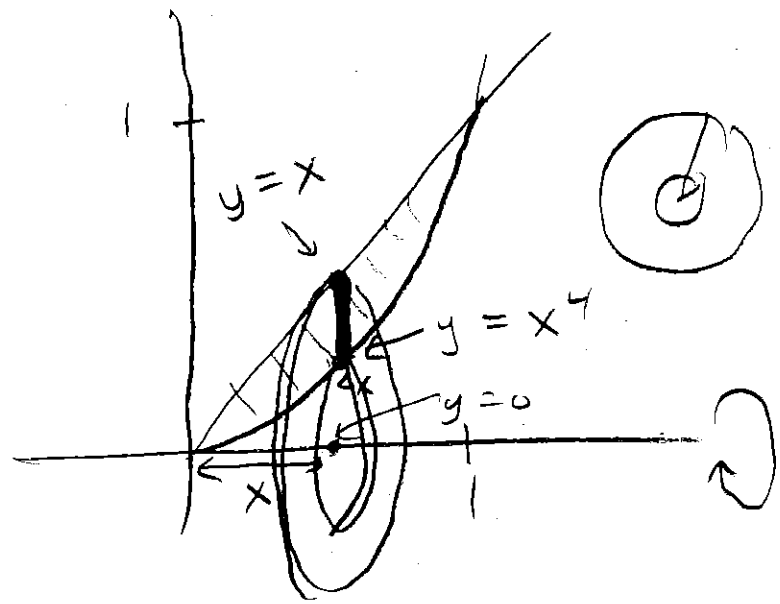


$$\int_0^1 \pi(x)^2 - \pi(x^4)^2 \, dx$$

$$\pi \int_0^1 x^2 - x^8 \, dx$$

$$\pi\left(\frac{1}{3}x^3 - \frac{1}{9}x^9\right)\Big|_0^1$$

$$\pi\left(\frac{1}{3} - \frac{1}{9}\right) = \boxed{\frac{2\pi}{9}}$$

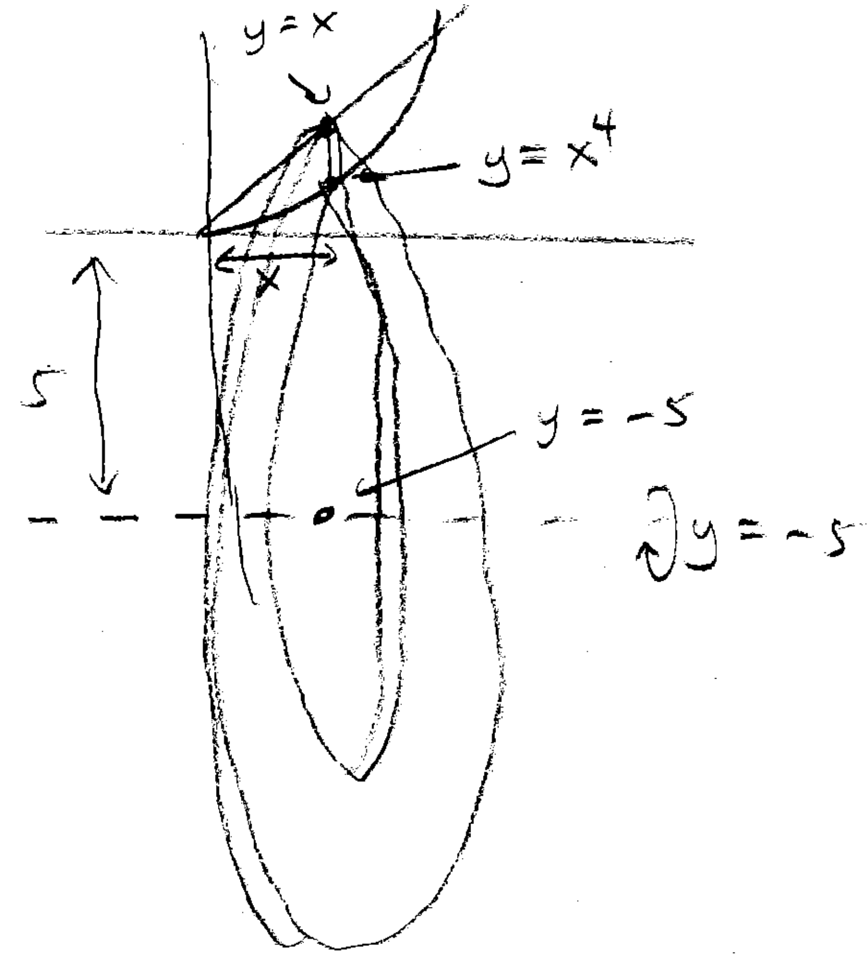


Example: Consider the region, *R*, bounded by $y = x$ and $y = x^4$. (*R* is the same as the last example).

(a) Now rotate about the horizontal line y = -5. What changes?

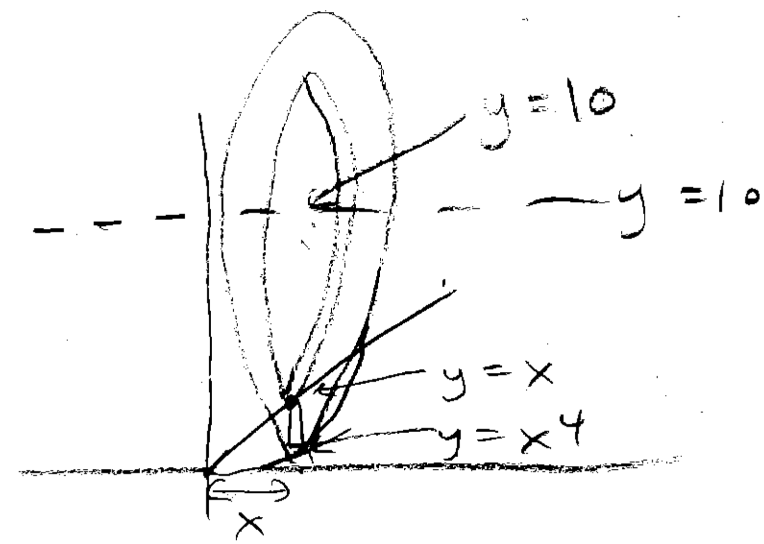
$$\int_0^1 \pi(x - -5)^2 - \pi(x^4 - -5)^2 \, dx$$

$$\pi \int_0^1 (x+5)^2 - (x^4+5)^2 \, dx$$



(b) Now rotate about the horizontal line y = 10. What changes?

$$\int_0^1 \pi(10 - x^4)^2 - \pi(10 - x)^2 \, dx$$



Example: $y = 2\sqrt{x}$ $\quad x = \left(\frac{y}{2}\right)^{1/3}$

Consider the region bounded by

$$4x = y^2 \text{ and } y = 2x^3.$$

Find the volume of the solid obtained by rotating this region about the y-axis.

$\boxed{\text{STEP 1}}$

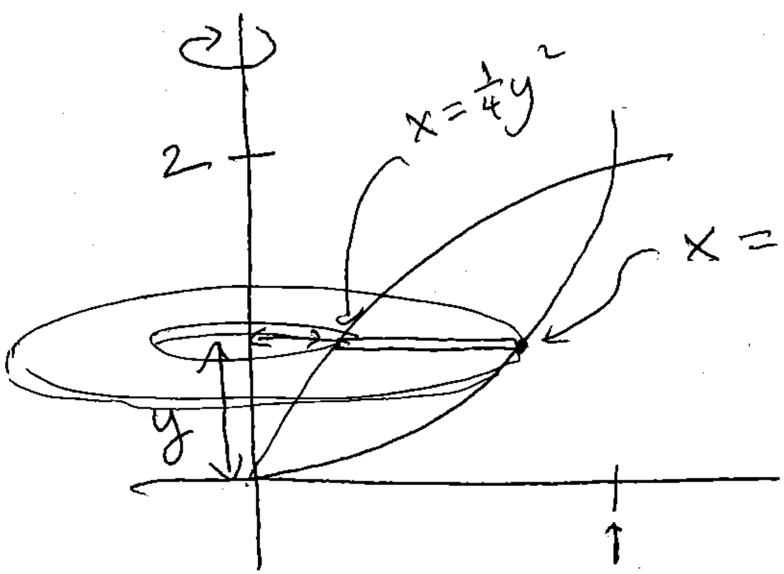
INTERSECTION:

$$4x \stackrel{?}{=} (2x^3)^2$$

$$4x = 4x^6 \Rightarrow x = 1 \text{ or } x = 0$$

$$\Downarrow \qquad \Downarrow$$

$$y = 2 \qquad y = 0$$



$\boxed{\text{STEP 2}}$

$$\int_0^2 \pi\left(\quad\right)^2 - \pi\left(\quad\right)^2 dy$$

$\boxed{\text{STEP 3}}$ FILL IN

$\left(\frac{y}{2}\right)^{1/3}$ $\qquad \frac{1}{4}y^2$

$$\Rightarrow \pi \int_0^2 \left(\frac{y}{2}\right)^{2/3} - \left(\frac{1}{4}y^2\right)^2 dy$$

$$\pi \int_0^2 \frac{y^{2/3}}{2^{2/3}} - \frac{1}{16}y^4\, dy$$

$$\pi \left[\frac{1}{2^{(2/3)}} \frac{3}{5} y^{5/3} - \frac{1}{16}\frac{1}{5}y^5 \Big|_0^2 \right]$$

$$\vdots$$

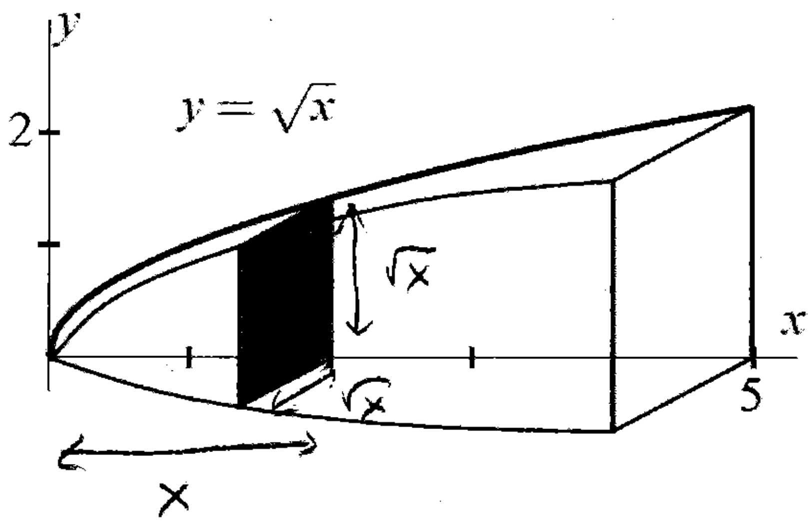
$$= \frac{4}{5}\pi \approx 2.5133$$

Example:

(From an old final and homework)

Find the volume of the solid shown.

The cross-sections are squares.



$$\int_0^5 (\text{HEIGHT})(\text{LENGTH})\;dx$$

$$\int_0^5 \sqrt{x}\;\;\sqrt{x}\;\;dx$$

$$\int_0^5 x\;dx = \frac{1}{2}x^2\Big|_0^5 = \frac{1}{2}25 = \boxed{\frac{25}{2}}$$

Summary (Cross-sectional slicing):

1. Draw Label
2. Cross-sectional area?
3. Integrate area.

This method has a major limitation:

6.2 method about *x-axis*, must use *dx*.

6.2 method about *y-axis*, must use *dy*.

What if the regions is rotated about the *x*-axis and we need to use *dy*?
(or about *y*-axis and we need dx?)

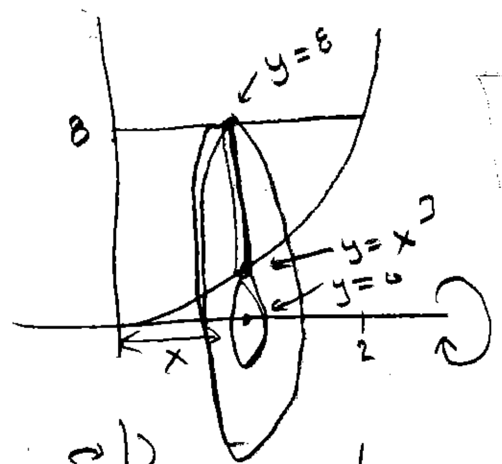
In these cases, 6.2 “Cross-sectional slicing” wouldn’t work!

We need another method.
That is what we will do in 6.3.

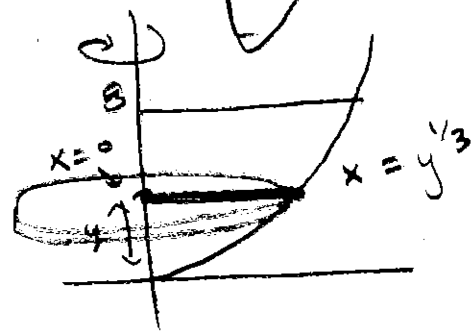
Close Wed: HW_3A,3B,3C
(complete sooner!)
Exam 1 is Thurs (4.9, 5.1-5.5, 6.1-6.3)

Entry Task:
Consider the region *R* bounded by
$y = x^3, y = 8$, and $x = 0$.
Set up the integrals that would give the volume of the solid obtained by rotating *R* about the ....

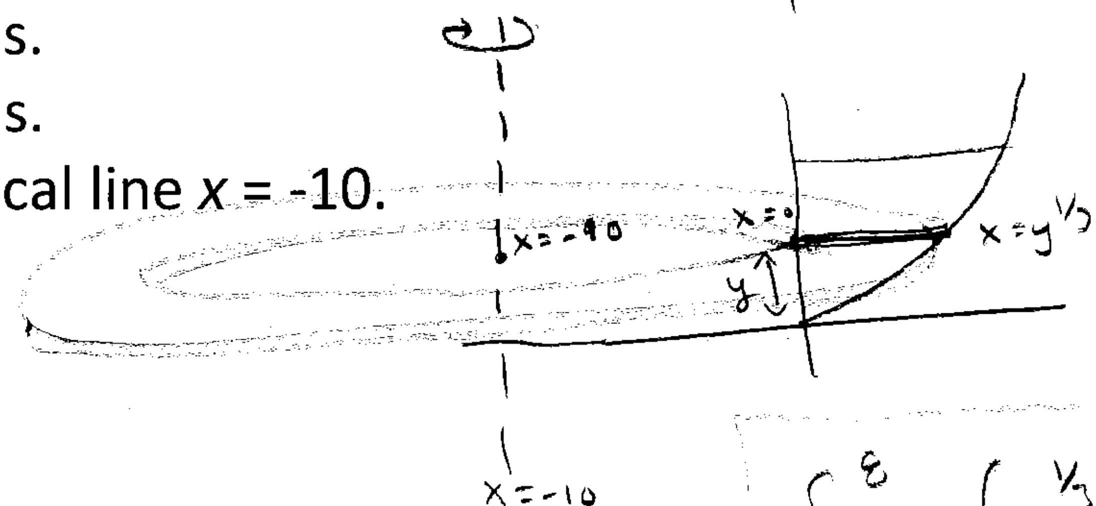
(a) ... *x*-axis.
(b) ... *y*-axis.
(c) ... vertical line *x* = -10.



$$\int_0^2 \pi(8)^2 - \pi(x^3)^2\,dx$$
$$\pi\int_0^2 64 - x^6\,dx$$



$$\int_0^8 \pi\,(y^{1/3})^2\,dy$$
$$= \pi\int_0^8 y^{2/3}\,dy$$



$$\int_0^8 \pi\,(y^{1/3}+10)^2 - \pi\,(10)^2\,dy$$